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REVIEW ARTICLE

See It as on Mathematical Thinking with Mathematical Representation: Mathematization in Mathematics Education for Human Character Formation

Masami Isoda

Bureau of Global Initiatives, University of Tsukuba, Japan
Email: isodamasami@yahoo.co.jp

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ABSTRACT
How can we configure the school mathematics curriculum to develop mathematical thinking? Theories in mathematics education explain it using different terminologies. Firstly, this article reviews theories and nature of curriculum for overcoming contradiction, secondly focuses on developing mathematical thinking in Southeast Asia Region for overcoming and focused on mathematical ways to see it as a part of human character formation, thirdly a theory of mathematical representations (Isoda, 1989 & 2015) is defined based on the ways to see it as, and fourthly mathematization process are explained by the representation theory for explaining how changes the way to see it as. These are illustrated by the cases of operational diagrams in figural representations, including Hanoi's Tower.

1. INTRODUCTION

How can we configure the school mathematics curriculum to develop mathematical thinking? To answer this question, we consider the objective of mathematics education as a part of human character formation and the configuration of the curriculum sequence to develop mathematical thinking. Making clear what mathematical thinking is necessary in the curriculum. The author focuses on mathematical thinking in the case of the ways to see it as, in other words a kind of student's belief, as a part of human character, and explains how we use it in mathematical representations. It explains the curriculum sequence for reinvention through Mathematization from the perspective of representations.

2. REVIEW OF CURRICULUM THEORIES TO DEVELOP MATHEMATICAL THINKING

Freudenthal (1973) explained it as Reinvention, characterized by the word 'mathematization' which means reorganization of experience. He also explained it as the objectification of lower-level methods of thinking by higher-level methods of thinking. In his didactical phenomenology (1983), Freudenthal claimed the necessity of making the mathematical entity describing the object of thinking clear before the psychological analysis of mathematical thinking and tried to list all mathematical terms' usages in schools. According to Freudenthal (1991), he also redefined mathematization as the progress of life in the world, in contrast to his followers' uncomfortable interpretations at his Institute.

Tall (2013) characterized learning mathematical thinking in the school curriculum by distinguishing three worlds of mathematics: Embodiment, Symbolism, and Formalism, and mutual transformations among them. Many of us believe that formal representations of mathematics, such as the worlds of Symbolism and Formalism, can be rendered in MathType within Microsoft Word. He also enhanced Figural and Manipulative Embodiment as mathematical representations. Indeed, in the history of mathematics education (Inprasitha et al., 2023), and so on, Figural representation has been essential, as has natural language, in Ancient Greece. When Arabic Arithmetic introduced

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Europe in Middle Age through Fibonacci's Libre Abaci (Singler, 2002), arithmetic used to be done by manipulative on roman abacus and it was replaced by Column methods on Arabic numerals from 1 to 9 which were called nine figures and 0 as Zepher instead of 'o', due to the symbols 1 to 9 on the Arabic alphabet did not existed on Roman Alphabet. Even Descartes enhanced Algebra, Arithmetic meant Column methods. Under the influence of Descartes's Universal Mathematics to establish Algebraic Form, Arithmetic expressions such as 1+1=2 were formalized as a part of algebraic notation (Inprasitha et al., 2023).

The Anthropological Theory of the Didactics (Chevallard, 2006; Winslow, 2011; Takeuchi & Shinno, 2020; Solis & Isoda, 2022) is explained by praxeological analysis with a quadruple (T, t, u, Q) consisting of a type of task T, technique τ, technology θ, and theory Θ. A couple (T, t) is called a practice block, a couple (u, Q) is called a theory block, and the whole quadruple (T, t, u, Q) is called a punctual organization. Since several practice blocks can be explained by a common technology, these groups of praxeologies are called a local organization, and the collection of praxeologies unified by a common theory is called a regional organization. From the perspective of Freudenthal's mathematization, the common theory of regional organizations does not constitute a general axiomatic theory in mathematics but rather a local theory. Local theories are not part of general theory because Anthropological theory was established in the context of the theorization of Didactical Transposition, which explains the difference between Mathematical Theory and School Curriculum.

Based on these theories, the curriculum can be configured like a net, with nodes representing local theories, such as curriculum standards' sentences, and connections that include contradictions among them in the the teaching sequence. Teaching-learning on the curriculum includes overcoming contradictions through the re-organizing process. In the Japanese National Curriculum Standards since 1968, the mathematics curriculum sequence is explained in terms of Extension and Integration, which support mathematization by reorganizing local theory. Local theories are not a subset of general theories. Among local theories, several contradictions exist. For example, on the extension of multiplication, the multiplication of whole numbers increases, but it is not true for decimals. The Japanese curriculum claims to develop mathematical thinking for overcoming contradictions, as well as necessary human competence, on the principles of Extension and Integration.

3. WAYS TO SEE IT AS A MATHEMATICAL THINKING CURRICULUM FRAMEWORK

In national curriculum standards across countries, there are general aims and issues, such as critical thinking and higher-order thinking skills, to develop human character traits necessary for the next generation's society, and to integrate them into every subject. However, it is not clear how to interpret them in the mathematics curriculum, because mathematics teachers focus on the knowledge and skills students should acquire, and the general aims for human character formation are stated in general terms that seem unrelated to specific knowledge and skills in mathematics. For bridging this divergence, the Southeast Asia Mathematics Curriculum Standards SEA-BES: CCRLS was developed (Isoda et al., 2024a). In the following, the ways to see it as' in mathematical thinking, which indicate a part of the human character, illustrate a bridge (see Hanson, 1965, 1969).

3.1. Mathematical ideas to fix the ways to see

On Figure 1, there are three ways to find the answer '27 tiles': Firstly, counting all for which we see it by each single tile, independently. Secondly, sums such as 10 + 9 + 8, which we see as distinguished by three groups of tiles. And thirdly, multiplying for which we move the top tile in the left group to the right group: Then, there are three sets of the center 9 blocks. Here, the tile in Figure 1 becomes manipulative beyond the fixed diagram.

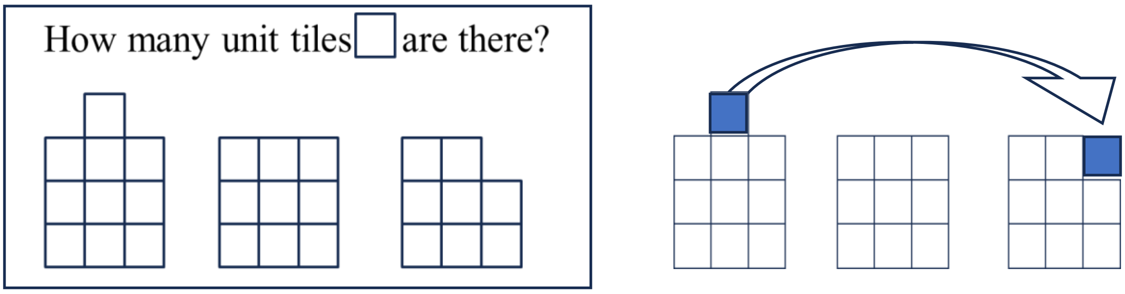


Figure 1. Isoda and Olfos (2021, pp. 124-126)

Here, the author calls it an operational diagram: To see Figure 1 in multiplication, we should develop students who can adapt multiplication by treating any number as the new unit for counting (a unit of measure). To move the tile, students must already understand the idea of multiplication, which finds the total amount by scaling the number of times using an unit of measurement to see 9 as a unit of scaling. As long as they want to apply the row of 9 on the multiplication table, which they already memorized, they do. It is a part of human competence.

Isoda and Olfos (2021), and Olfos et al. (2025) demonstrated that students change the way they see the figure for the same 'how-many' question through learning addition and multiplication from first to third grade in Japan. It implies that we should teach necessary mathematical ideas when we teach mathematical knowledge and skills, because these ideas include ways to see the world for the significant use of corresponding knowledge. In Figure 1, if students learn to see that any number can be a unit for counting, as well as memorize every row of multiplication, they can move. Indeed, Japanese students can move because it is included in their textbook as part of the curriculum. If teachers can't recognize it as content and skip it, students may fail to move. For teacher training, SEAMEO RECSAM, SEAMEO STEM-ED, and UT-CRICED prepared several guidebooks for teachers who are unable to access the task sequence of English editions of Japanese Mathematics Textbooks (Isoda et al., 2024b; Teh et al., 2024; Gan et al., 2024d; Somsaman et al., 2024). In traditional terminology in mathematics education for explaining cognitive process since the 1980s, students' beliefs are used to explain the source of every student's perspective, the way they see, which is emergent in class or in everyone's problem-solving, and observers explain it as the existence of students' beliefs (see García & Dolores, 2021). For observers, the origins of students' beliefs and what students learned before are beyond their reach because they cannot follow the long-term learning trajectory. Thus, it is emergent for them. However, from the perspective of teachers or curriculum designers, mathematical ideas are part of the objectives that they should teach in schools. Especially in the case of the Japanese mathematics curriculum, mathematical thinking has been enhanced since 1951, and Japanese researchers theorized it in several ways (see, for example, Isoda & Katagiri, 2012). Current mathematics textbooks in Japan, such as Gakkotosho (see the English Editions, Isoda & Tall, 2019; Isoda & Kusaka, 2024), are used in teaching and employ understandable language for teachers and students. It is the background that allows Japanese students to see it as.

3.2. Structure of Mathematical Thinking on the Curriculum Framework and See it As

On Figure 2, the curriculum framework of the Southeast Asian Basic Education Standards (SEA-BES) consists of three main components: (I) Value, Attitude, and Habit; (II) Ideas, Ways of Thinking, and Activities; (III) Contents (Isoda et al., 2024; Montecillo et al., 2018). In this context, 'Value' refers to fundamental beliefs or ideals important for human character, 'Attitude' means a settled way of thinking or feeling, and 'Habit' represents regular practices. 'Ideas,' 'Ways of Thinking,' and 'Activities' refer to approaches and actions involved in the thinking process. 'Contents' focuses on mathematical subject matter. All these components together define mathematical thinking within the framework. Each is necessary for human characterization.

Every component in (I) and (II) is illustrated with example terminology that can be replaced with contextual, meaningful, and understandable words or sentences, as seen in Gakkotosho textbooks. For this framework, 'terminology' refers to representative words that capture essential educational ideas or processes. The layers within the framework describe the objectives of mathematics teaching in every class. Every standard sentence in (III) specifies particular content to be taught in the mathematics curriculum sequence. In (III), each 'key Stage' refers to a developmental period, and 'strands' are thematic groupings of standards that explain the focus rather than being fixed domains. Each 'standard,' here considered a local theory, can include some contradictions with other standards; the order of standards shows a progressive sequence of curriculum development in relation to mathematical content.

For example, within (III) (Contents), Figure 3 shows the curriculum sequence for multiplication (Isoda & Olfos, 2021). Each 'box' in this figure represents a local mathematical theory - a specific conceptual approach - that functions without contradiction within itself, though contradictions may arise between different local theories. Each 'sub-box' provides examples of such contradictions, which may cause confusion for students. For reference, 'local theory' here denotes a specific, often nuanced, viewpoint or rule within mathematics. Well-learned teachers and mathematicians may not perceive contradictions, but students encounter difficulties when applying learned concepts due to apparent inconsistencies.

Components of (I) and (II) in Figure 2 are general contents of teaching which can be applied to any subjects beyond mathematics, as long as words are replaced by corresponding appropriate words in every subject. For example, 'beautifulness' as the value in (I), 'pose question' as attitude in (I), 'recursion' as idea in (II), and 'Inductive' as thinking in (II), the entity for these words can be discussed in any subject's teachers, such as the art teachers' beautifulness, even the entity of the word is different depending on the subject. And if we fix the specific content within a subject, it becomes meaningful within that subject as an entity. (I) and (II) are learned through the appropriate process for specific content in (III). The appropriate learning process in the curriculum includes the moment of reflection and appreciation to recognize (I) and (II). In Education, Mathematics is the subject for developing human character (I) and (II), with the specific content of mathematics in (III).

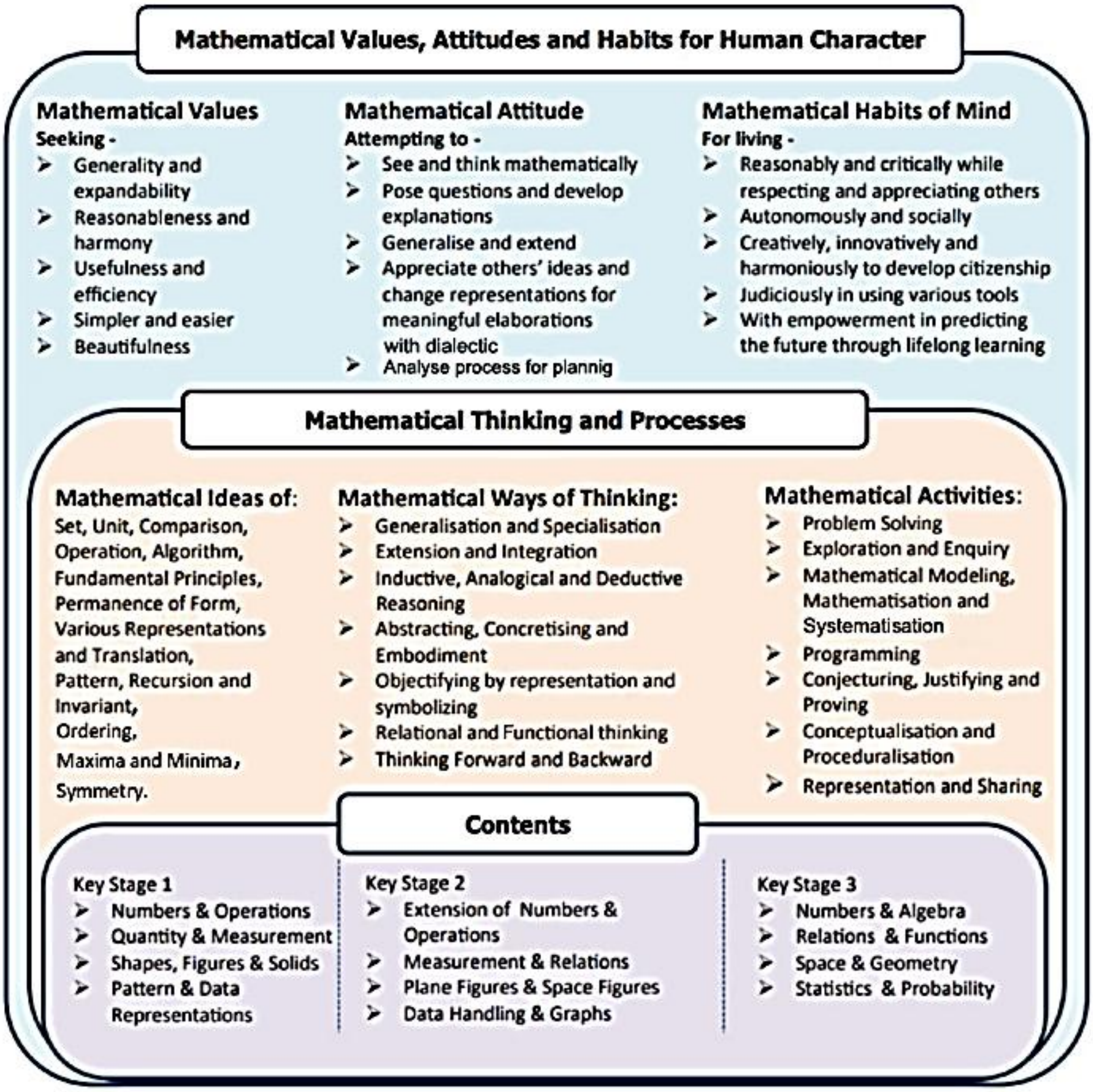


Figure 2. Second Edition of SEA-BES: CCRLS Framework (Isoda et al., 2024a, p. 8)

Unfortunately, in many cases, in lesson plans, teachers only write the objective for specific content, especially knowledge and skills under (III), and never write objectives for (I) and (II). If we ignore the objectives on (I) and (II), the exercise in (III) for skills looks like the best way to teach. Even in exercise for III, if teachers consider (I) and (II) as for human character formation, an appropriate task sequence that provides opportunities for reflection and appreciation for the exercise to develop (I) and (II) is feasible. Today, (I) and (II) are called Generic Competence and

(III) is called Specific Competence instead of Ability. Due to the term “ability” carrying with it a subject-segmented ideology, like teachers who only write the objective on (III) and inconsiderable (I) and (II).

Figure 2 is explained analytically by indicating three Areas for ‘See As’, ‘Ways of Thinking’, and ‘Habit’ (Figure 5). It explains as follows: The area for See As in Figure 5 is related to the values in (I), the ideas in (II), and all the content in (III). Value and ideas are used to explain the ways to see. Both are Generic Competence for See It As. In relation to the content as specific competence, there are content-specific ideas to see as explained by the example of multiplication in 3.1 on Figure 1. The content-specific ideas in (III) provide ways to see only limited functioning within the local theory, which makes it possible to use ideas such as Figures 3, 4a, and 4b. The area for Ways of Thinking on Figure 5 relates to Attitude in (I) and Ways of Thinking (II) for generic competence. Each piece of content in (III) also has a specific way of thinking. The area for Habit on Figure 5 is related to forms of activity such as Habit in (I), various types of mathematical activity in (II), and each content, such as automatized skills and procedures.

The SEA-BES curriculum framework (CCRLS, Figure 2) serves as the basis for every standard in (III) to clarify the objectives and teaching sequences for each class. It is a set of terminology for lesson study that explains students’ activities and understanding.

On the other hand, in mathematics education research, the term ‘belief’ describes students’ cognitive processes as part of their identity. From an educational perspective, it is a human trait that teachers can develop. When we compare the Figure 2 curriculum framework with the belief in cognitive research, a major difference is that teaching in the curriculum of Figure 2 involves specifying class objectives and planning instruction based on what students have learned before, which was also taught by the teachers. If teaching (I) and (II) under the teaching of (III) are done for the preparation of future learning, students are able to learn mathematics by and for themselves through thinking mathematically. For example, in Figure 1, if students learned the idea of multiplication and memorized Row 9 of multiplication, they may be able to move on and appreciate the significance of multiplication, which makes the situation simpler and easier to understand in terms of value in (I). It looks like a hypothesis, but Japanese students can learn from their textbook (Isoda & Olfos, 2021) and grow through its curriculum implementation.

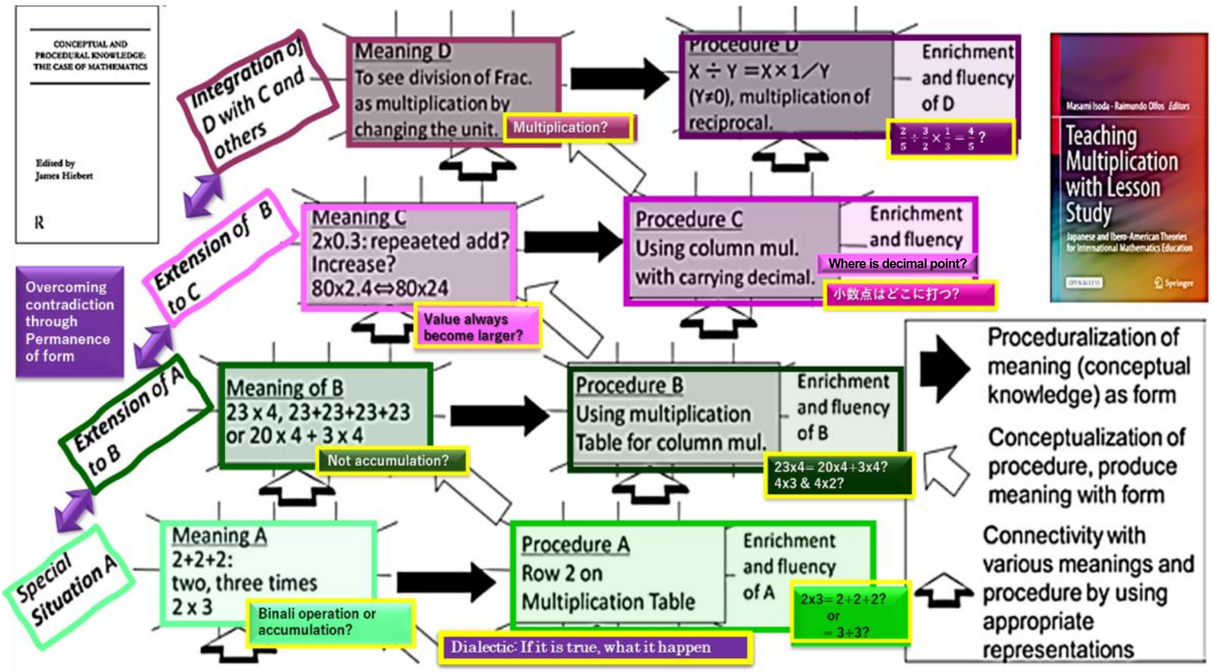
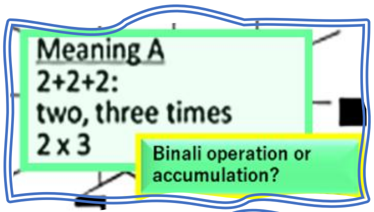
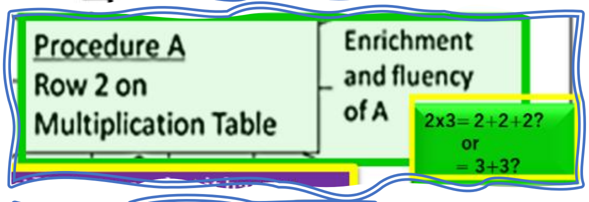


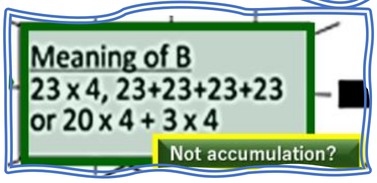
Figure 3. Local theory sequence through extensions (Isoda & Olfos, 2021, p. 8, revised)



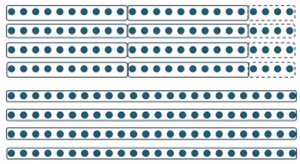
Answer of Multiplication is given by accumulation as adding the same number however it becomes the new operation by seeing any number as the new unit for counting (unit for measure) (?).
Addition was not only adding the same number but also different numbers.



On real situation, two cars of four wheels and four bicycles of two wheels should be distinguished. Both cases, there are eight wheels (?).
In mathematics, multiplication is a commutative binary operation on the field.

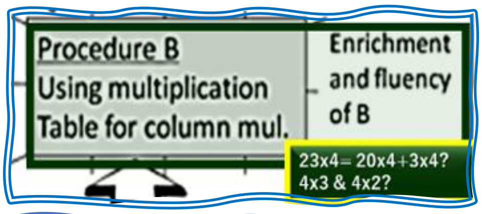


On the base ten place value system, see it as

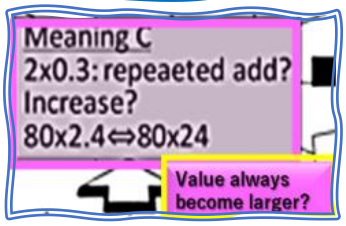


instead of

(for what?)

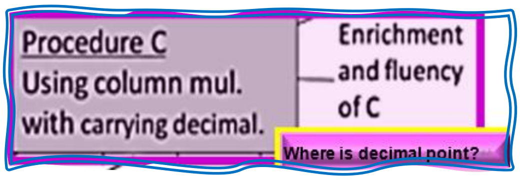


On the column multiplication, instead of saying twenty times four, we say 4 times 3 for applying the row 4 on multiplication table (?).



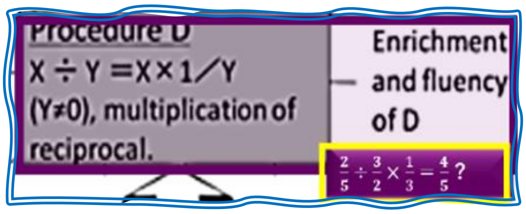
How many times shall we repeat when the multiplier is decimals on real situation?
The products in multiplication of natural numbers always increase. However, does it increase on the multiplication of decimals?

Figure 4a. Explanation of Contradiction on Sub-boxes: Different ways to see it as



On the column addition and subtraction, digits should be kept the same rows on the same place values on the base ten system, and the positions of decimal points never change. However, we break the rule and must tack care the position of decimal point (?).

Division used to be the inverse operation of multiplication. However, on the division of fraction, it becomes multiplication of reciprocal fraction. Is division multiplication?



Additions or multiplications of three whole numbers are commutative and associative, but how about subtractions or divisions?
Now division of fractions are multiplication (of the reciprocal). Then, commutative or associative?

Figure 4b. Explanation of Contradiction on Sub-boxes: Different ways to see it as

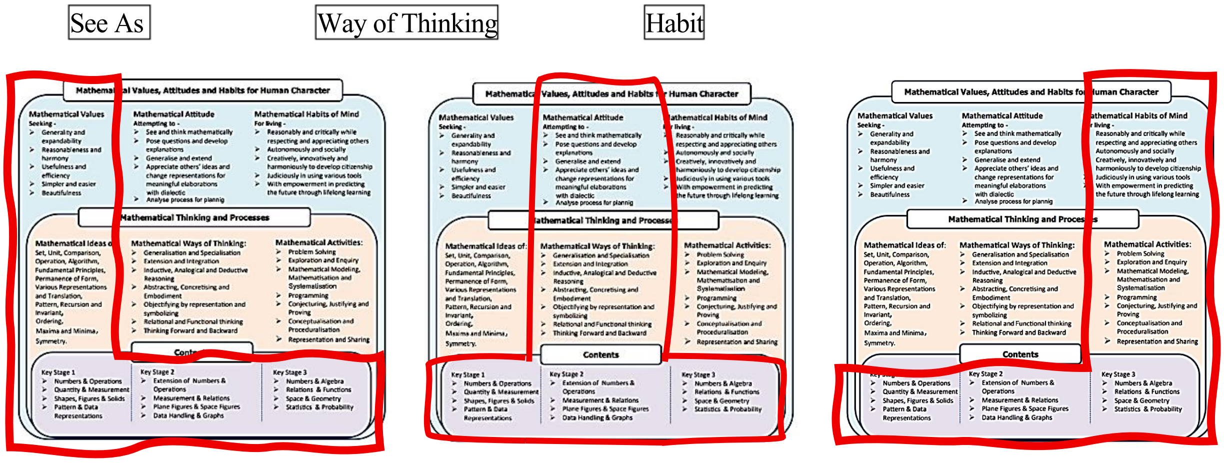


Figure 5. Three Arias for See As, Ways of Thinking and Habit on Figure 2

4. Theory for Mathematical Representations on the way to see it as

To generalize Freudenthal's mathematization, the author (Isoda, 2015, originally in 1989; Isoda, 2018) theorized the nature of Mathematical Representation through the case of Algebraic Representation, focusing on how it can be seen. It makes clear the content of teaching, which Freudenthal explained using the term 'entity,' and then psychologically explains the emergence of students in the context of observers.

4.1. Objective and Context: Ways to see it as

Mathematical representations convey objectives and contexts by using specific, sequential forms and by interpreting mathematical sentences when the forms are understandable. It can be used in specific ways to see. Here, Figure 6 provides an example of different ways to view what is mentioned in Figure 5 in the context of Figure 2.

Q1. What is it? $x = 3$ $2(x-1) = 4$ $2x = 6$	Q2. What is it? $2(x-1) = 4$ $2x = 6$ $x = 3$	A1. $2(x-1) = 4$ $2x - 2 = 4$ $2x = 6$ $x = 3$	Q3, When $2(x-1) = 4$, Find the value of $\frac{x-1}{2}$

Figure 6. Various ways to see it as for equation

In Q1 of Figure 6, one interpretation is a family of equations where each solution is 3 - that is, "a set of equations whose x-value is 3," or "if we solve each equation, the answer is 3." If we see them as a set from this perspective, it can be seen as a family of equations. Otherwise, it is simply different equations.

In Q2 of Figure 6, the possible interpretation is "It is solving the equation $2(x-1) = 4$" (see Q2 and A1), or, if explained step by step: "from the first line to the second, expand and transpose; from the second to the third, divide both sides by 2" (see A1). In mathematics, the interpretation of Q2 as sequence of equations is 'the solving an equation', which is the technical term for this kind of sequential representation of equations which indicate the ways to see in synthesis as whole for the Gestalt view. At the same time, the term 'solve the equation" initiate some algorithm which meant a Way of Thinking and Habit (named specific process) at Figure 5 in the content on Figure 2. Thus, it specifies both the objective and the context. On these discussions, if we see only "$x = 3$" as a single form, we cannot determine the objective without adding a sequence of equations. Indeed, "$x = 3$" can indicate different contexts, such as "substitute $x = 3$ into other equations in Q1" or "the solution is $x = 3$" in Q2. In contrast, if an interpretable set of sequential forms by mathematical terms is provided, we can uniquely specify both the objective and the context.

In Q3 of Figure 6 (see such as Morozumi, 2004), a mathematical problem is presented in a sentence that includes both an equation and an expression. There are at least two possible approaches: The first approach for finding the value of $\frac{x-1}{2}$, we first need to know the value of x, so we solve $2(x-1)=4$ and then substitute the answer to $\frac{x-1}{2}$. The second approach is to notice that the problem contains the same term, $x-1$, twice. From this, first, find $x-1=2$, and then substitute it directly into $\frac{x-1}{2}$. Two approaches illustrate the different ways to see it as.

As Polya (1945) pointed out, a problem is divided into sub-problems to find a solution. The entire solution is usually summarized in a brief statement. To explain it further, terms for each sub-solution become necessary and are summarized in a statement. These statements explain why certain steps in the solution are necessary. Furthermore, each sub-solution is divided into several steps. Although each step is explained in terms of its operations, these only indicate the logical correctness between lines. Even if we accumulate the sequence of all such lowest steps, it provides only logical correctness but does not necessarily reveal the objective and context - that is, the Gestalt view of why we are performing them in order to solve the problem. In this hierarchical structure, the whole solution and its sub-solutions are interpreted in terms of why, which explains the objective, while the individual steps are interpreted in terms of logical correctness.

4.2. Focused Symbols and Operations

Freudenthal (1983) analyzed the terminology used in school mathematics to clarify the mathematical entities that indicate the objects of thought. The layers, which are explained as a hierarchical structure, are also analyzed with Peirce's semiotics, which explains how the meaning of a sign is fixed by the representamen (sign), the semiotic object, and the interpretant (Atkin, 2023).

In mathematical problem solving, the representamen are specified at several layers, such as the entire solution as the top layer, the sub-solution as the second layer, and the individual steps as the third layer, at least in Q3 in Figure 6. The semiotic object for interpretation changes across these layers, and the mathematical terminology Freudenthal mentioned is used to distinguish among them.

From the perspective of the objective in solving, symbols and operations as objects of thinking are fixed depending on how we interpret the problem and solutions. For example, in the first solution of Q3, the focused symbol is x, and the focused operations are solving for x using the properties of equations. In the second solution, the focused symbol is '$x-1$', and the focused operation is substitution of the value '$x-1$'. In mathematical terminology, in this case, the expression $x-1$ is treated as 'a value' instead of operation - reflecting a way to see the expression and equation as a Gestalt.

These ways to see provided the object of thinking. Thus, teachers are necessary to make clear when they teach Q3 and necessary to provide the opportunity to reflect and appreciate how to see the expression as the value is significant.

4.3. Definition: Mathematical Representation

A unit of mathematical representation, as a specific set of sequential forms, is defined by its objective, focused symbols, and focused operations. Focus is fixed by the objective, and the objective specifies the ways to see symbols and operations, or 'ways to see' specifies the objective, on the contrary. It explains the significance of the unit of representation.

It can be mathematically interpreted in terms of why (solving the equation as in Q2), which secures the idea for reasoning, while the forms in between are explained by operations (logical steps as in A1), which secure correctness. In the Q2 sequence of equations, the focused operation refers to the algorithm for solving equations, even though each step also involves operations. Thus, the unit of representation that conveys the objective is centered on mathematical representation as a set of sequential forms rather than on each individual form. Please note again: on this definition, two types of answers for why questioning are distinguished (Isoda & Katagiri, 2012, pp.3-10, pp.127-128). One answer is the Gestalt view, which explains significance and offeoffersion; it also provides some symbols and operations, as well as the way to see it. Anwell aser answer for why a step-by-step approach is a logical correctness, which does not refer to this definition.

According to Peirce's semiotics, a sign is not limited to symbols but can be anything. Extending this framework, when we broaden mathematical representation beyond algebraic forms to include figural representations, we consider 2-D diagrams as symbols. Since the signs of semiotics are unusually associated with operations, we retain the word symbol as the object of operation instead of the sign. Within this extension, drawing and manipulation can also be regarded as operations (Piaget, 1972). It is illustrated in the next chapter in the case of diagrams.

5. EXTENSION TO FIGURAL REPRESENTATION

In this section, the application of the theory in the context of figural representation is demonstrated in Figure 7 (Isoda, 2015).

5.1. A Case Study

Isoda (2015) posed question Q4 to clarify the significance of figural representation to a graduate student who wishes to study the function of figural representations in algebra.

The following is a summary of the discussion:

Firstly, on the board and without using a diagram, she solved the problem by deducing an equation with the length set as x (A1 in Figure 2). She obtained $x = \frac{5}{3}$ as the length, but then tried to erase it. When Isoda asked why, she explained that the fraction seemed strange for this problem. Secondly, she re-deduced the equation and obtained a negative width. At that point, she began to doubt the original problem and reconsidered her deduction (A2). Thirdly, she redefined x as the width, re-deduced the equation, and obtained 14 cm as the width (A3).

After confirming her answer, Isoda asked her to solve the problem using a diagram. She then sketched the problem situation as shown in Figure 7. When Isoda asked her the significance of her drawing, she replied that it was useful for deducing the equation without overlooking "twice," "double," or "two times." Isoda agreed, telling her that her drawing was a sketch, and discussed that it is usually used by teachers as scaffolding to explain how to deduce equations from original problem sentences. At the initial stage of equations, students draw diagrams, but later they do not feel the need to draw because they can deduce without drawing. It functions for scaffolding.

Q4. Problem.

There is a rectangle whose length is 3 cm longer than width. A rectangle is drowned by doubling the width of this rectangle and tripling its length. The perimeter of the resulting rectangle is 10 cm longer than twice the perimeter of the original rectangle. Find the length of the perimeter of the original rectangle.

(It was posed by only sentence: from the lecture of Hiroshi Nemoto, Inspector of MOE, Japan)

A1:

length x, width $x - 3$

$3x$ $\quad$ $2(x-3)$

$2\big(3x + 2(x-3)\big) = 2(x-3) + 10$

…omitted…

$x = \frac{5}{3}$

A2:

length x, width $x - 3$

$3x$ $\quad$ $2(x-3)$

$2(5x-3) = 2(2(2x+3) + 10$

…omitted…

$x = 2$

A3:

length $x + 3$, width x

length $3x + 9$, width $2x$

$2(5x+9) = 2\big(2(2x+3)\big) + 10$

…omitted…

$x = 2$, width 2cm, length 5cm

Answer. 14cm

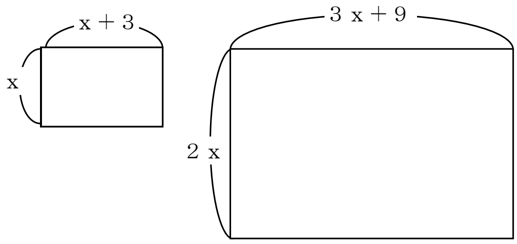
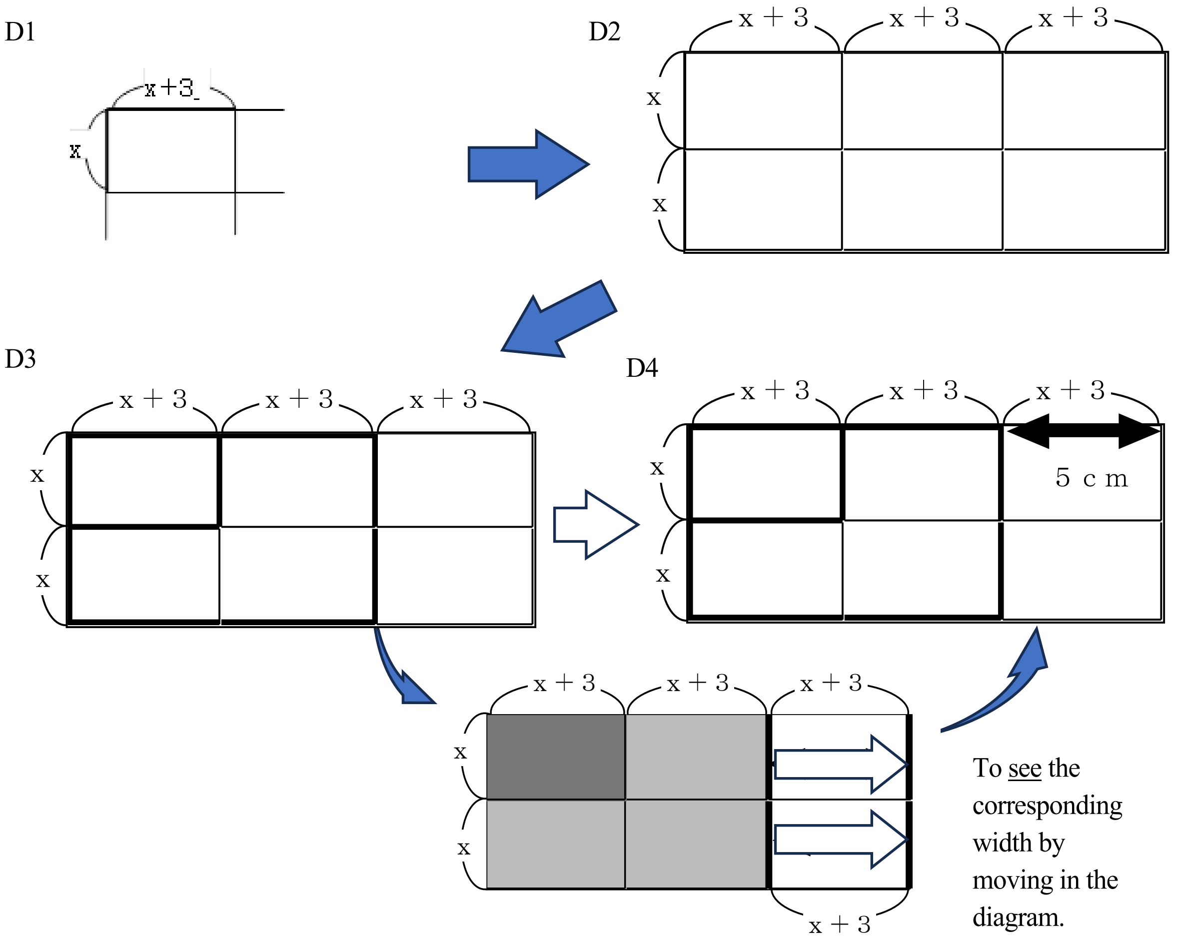


Figure 7. Is the answer, fraction, or negative?

After the discussion, Isoda asked her to solve the problem again by drawing diagrams that faithfully represent each sentence of the problem. She did not understand this order and could not proceed. Then, Isoda instructed her to draw the diagram beginning from D1 in Figure 8 for the first sentence, "There is a rectangle whose length is 3 cm longer than its width' in Q4 of Figure 7. And asked her to continue drawing D2 for ‘A rectangle is drowned by doubling the width of this rectangle and tripling its length’.



Problem’ for Generalization
There is a rectangle whose length is □ cm longer than width. A rectangle is drowned by □ the width of this rectangle and □ its length. The perimeter of the resulting rectangle is □ cm longer than □ the perimeter of the original rectangle. Find the length of the perimeter of the original rectangle.

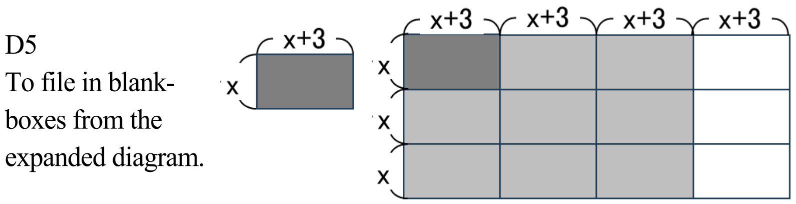


Figure 8. Solution by Diagram

However, she could not go further in interpreting the diagram which she had drawn for the sentence ‘The perimeter of the resulting rectangle is 10 cm longer than twice the perimeter of the original rectangle’. Isoda asked to confirm which rectangle in the diagram represented the doubling of the original rectangle's length and width. She marked D3 but did not go further. Then Isoda asked where the 10 cm measurement appeared in the diagram. By moving the width, she identified the corresponding length and obtained D4.

After the session, Isoda reflected from D1 to D4 and asked about the further significance of drawing diagrams. She noted that the problem can be solved not only with equations but also with diagrams. And Isoda instructed her that extended problems in Figure 8 can be more easily developed if we understand the structure of this problem through D4 and its correspondences.

In Problem' for Generalization shown in Figure 8, if we insert different numbers into the boxes, we can create new problems. However, we cannot be certain whether these problems are well-posed or ill-posed until we deduce and solve the corresponding equations. If we draw an expanded diagram like D5 based on the idea in D4, we can pose new problems whose solutions are already known. In this way, it becomes possible to provide general solutions for this type of problem, which was developed in the manner of D5 based on the unit diagram D1. The solutions by diagrams in Figure 8 provide the local theory for a set of meaningful problems. Even though the algebraic solution by equations is general, this figural solution is much more significant and stronger in a specific situation locally. Here, the ways to see diagrams and the ways to see the problem are extremely different.

5.2. Operational Diagram as Mathematical Representation

According to the definition of the unit of mathematical representation, we can distinguish between a sketch such as that in Figure 7, which supports deducing an equation from the problem situation, and the diagrams in Figure 8, which provide a general solution for this problem type (e.g., D5) without deducing equations. The diagrams in Figure 8 form a sequence whose objective is to solve the problem diagrammatically (objective) by arranging the unit rectangle (focused symbol) to generate other rectangles and by shifting widths or lengths to corresponding positions (focused operation). Through these objective, focused symbols and focused operations, the sequence of diagrams constitutes a unit of mathematical representation.

In contrast, the sketch in Figure 7 was not treated for the object of operation and was unnecessary. Indeed, the student reached the correct answer in Figure 7 before drawing, suggesting prior training in equation-based problem solving without diagrams.

On this basis, Isoda defined this kind of diagram, which functions as a mathematical representation, as 'operational diagrams'. The operational diagram has operations (manipulations) to change the diagram (as a symbol). The problem can be solved using the operational diagram. In Figure 8, arrows indicate the operations to change the diagram. It represents the thinking process step by step. Operational diagrams are distinguished from diagrams that teachers use only for the purpose of explanation, as opposed to scaffolding. Operational diagrams follow the rules of drawing, which specify the operations (manipulations) for changing diagrams, which students implement to achieve a specific objective in problem solving. Operational diagrams define the local theory for a specific solution, such as D5 in Figure 8.

In Figure 1, the idea of multiplication is used to see the fixed diagram as an operational diagram. In Figure 8, if students grasp the idea of the correspondence of sides, they can meaningfully extend it to a special type of mathematical problem. An operational diagram provides specific ways to see the diagram 'operationally': in the case of physical objects such as blocks, it implies the manipulative blocks when meaningful ways of manipulating them are fixed.

6. Theory of Representation for Mathematization

To generalize Freudenthal's curriculum theory, Isoda (2015) developed a representation theory based on Mathematization.

A unit of mathematical representation, defined as a specific set of sequential forms, consists of an objective (Ob), focused symbols (S), and focused operations (Op). This unit is expressed as Ob(S, Op). The objective determines the focus. Using this notation, Figure 9 summarizes the process of mathematization, illustrating the progression from local theory in Figure 7 to further local theory in Figure 8. In Figure 7, the original objective is 'Solve Q4 by equation on unknown x,' or simply, 'find x'. The focused symbol is 'equation' for Q4, and the focused operation is 'solving' equations, summarized as 'Find x (Equ., Solv.)'. In A1, A2, and A3 of Figure 7, different equations are formulated, so Equations 1, 2, and 3 are distinguished. From D1 to D3 in Figure 8, each sentence in Q4 is represented using the unit rectangle D1. At the beginning of D1, the objective is 'drawing rectangles (Dr1 in Figure 9)'; however, this does not include finding the solution by diagram. In D2, the objective is 'to use a unit rectangle for drawing (Dr2)'. From D3 to D4, the objective changes to 'finding x in diagrams (Dr4)': by observing the width as the sides of the diagram are moved, the answer is found by arithmetic.

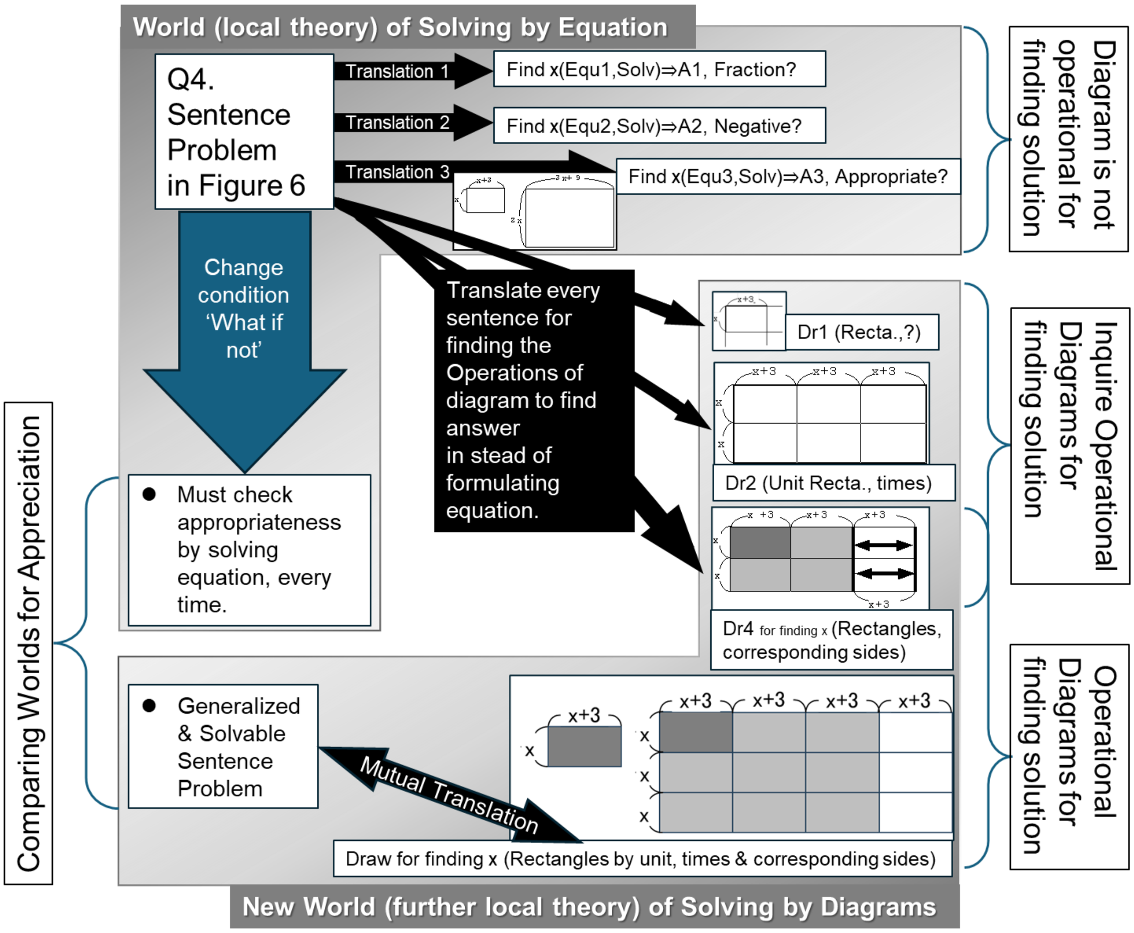


Figure 9. Mathematization from the perspective of representations on Figure 7 and Figure 8

In Figure 9, black allows the translations between different representations. These allow implicating what is necessary for mathematization from local theory to further local theory from the perspective of representations. In the First Stage, there is a local theory to solve a set of problem situations. There are further symbols which emerge for the translation of their solutions. One of them becomes an alternative symbol in later theory; however, it is not yet a focus in current theory because it lacks an operation for self-generation, which supports further theory. In the Second Stage, to further develop the theory, the alternative symbols require new operations that enable self-generation. Through translations from original situations, new operations for new symbols are established. Once new symbols and operations are confirmed to generate a set of problem situations that can be solved by them, they are recognized as a further theory, making it unnecessary to use the old theory to solve them. In the Third Stage, there is an an alternative theory that solves a set of specific problems by introducing new representations, symbols, and operations.

7. Mathematization in Hanoi's Tower

Let's analyze the process of mathematization in Hanoi's Tower through the lens of representation theory, drawing on my experiences in workshops on computational thinking (Isoda et al., 2021; Somsaman et al., 2024).

7.1. Solving Hanoi's Tower

Here, I would like to illustrate the case of undergraduate students in Thailand.

First, I posed the Hanoi Tower Problem using a concrete model with three rings**.** Students solve it and understand the rules: 1) There are three poles, and there are three rings in each pole whose size gets larger from top to bottom. 2) A rule is that only one ring moves to another pole at a time. 3) Another rule is that the larger ring cannot move on the smaller ring. 4) On this rule, move all rings from the first pole to another pole. I asked how many times you can move the rings in the minimum number of moves without going back. They did it again, and I define it as 'step'. And then I ask them to solve the four-ring case for counting steps. Some of them began to stop when they moved the largest rings after they moved three rings to another pole. Because they recognized it was troublesome to do the same things in the case of three rings, again. After some of them recognized it, I called them to the front, and they explained why they stopped by using the term 'repetition' or 'have to do the same as the case of three, again'. (until here, around 15 minutes).

Secondly, to confirm the repetition, I asked them to solve the problem by drawing a diagram. Many began with the case of three or four rings; a few started with one ring. When I found someone who started with one ring, I informed them and notified others. If no one did, I asked them to draw on the One Ring case.

Many of them did not draw the diagrams in a well-organized form to show repetitions at a glance, and some skipped drawings in some steps. I asked why they would like to skip. I confirmed that each step is done under the rule and in no other way. When all students began drawing the case of three (until here, around 35 minutes in total), I shared the following step-by-step instructions up to the case of four rings (Figure 10) and asked them what repetition is (or what it means to do the same things as in the past).

Students found the following:

i. Tower Diagrams for the case of some rings are twice the duplicate diagrams of the previous case. It looks like a copy-and-paste if we ignore the largest ring.

ii. The number of diagrams becomes twice that of the previous case because if we ignore the largest ring, it is a duplicate of the previous case.

iii. The additional step is to move the largest one.

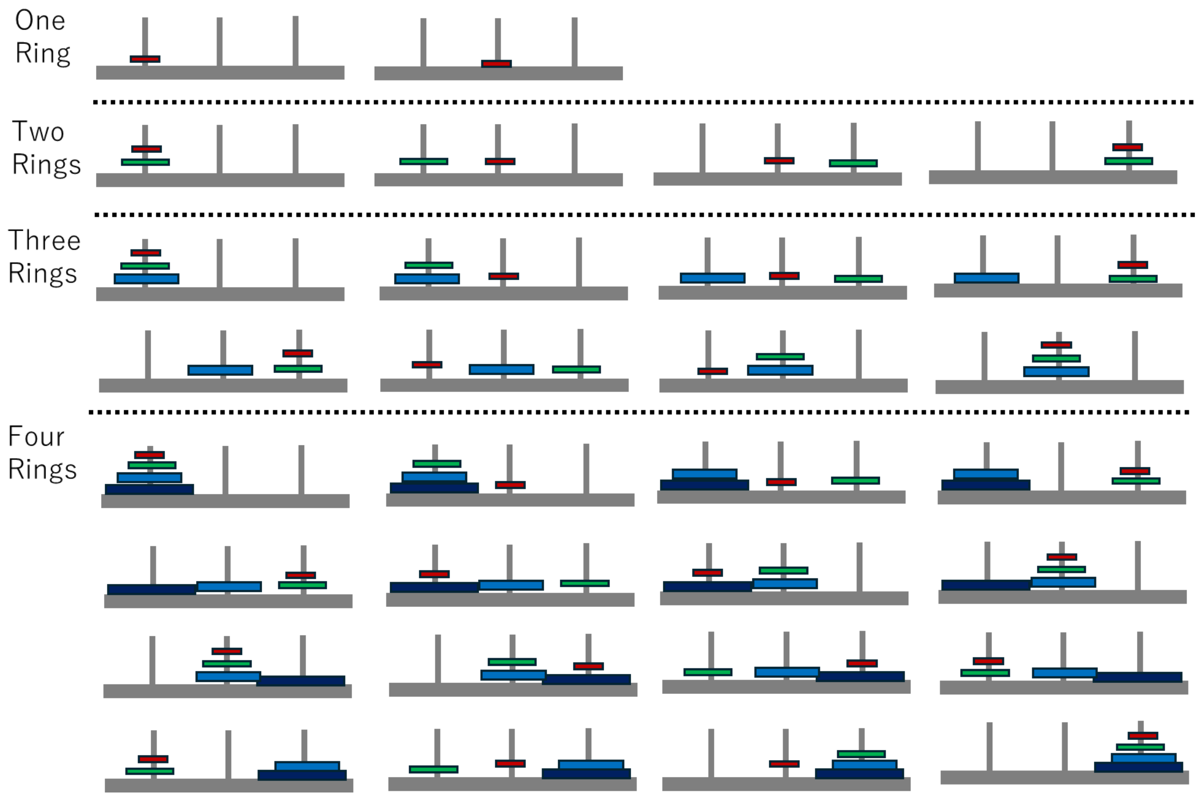


Figure 10. Operational diagram for Hanoi's Tower to illustrate recursion

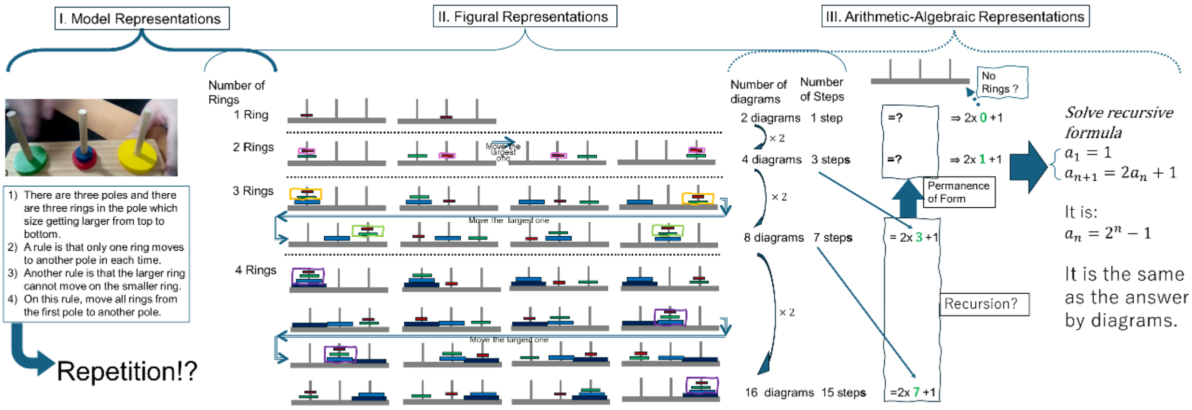


Figure 11. Solution of Hanoi's Tower in Several Representations

Finally, they solve it algebraically (until here, around 50 minutes).

I summarized their findings in Figure 11.

And explained to them what the idea of recursion on mathematical ideas in Figure 11 is. They recognized it as follows:

(I) On model representation, it is embedded the feeling of being troubled to repeat the same things.

(II) On figural representation, it is twice the duplicates of the previous case if we ignore the largest ring. It looks like a copy-and-paste of the previous case. And moving the largest ring is one step.

(III) On arithmetic-algebraic representations, it is twice the previous value plus one.

I confirmed each of them as instances of recursion, which was illustrated in distinct ways depending on the representations. In the model, students found the repetition troublesome at 4 rings but did not see it as a duplicate. On figures in II, it was twice duplicated, but in the case of no rings. Some students could not draw every step exactly, but skipped steps by treating repetition as troublesome. In arithmetic-algebra in III, the formula is given for recursion, and the case of no rings is also considered under the permanence of form. These differences illustrate three theories that include minor contradictions with each other. Indeed, between I and II, if we skip drawings, the number of diagrams cannot be seen. Between II and III, the case of no rings is considered in III but not in II.

7.2. Illustration of the Mathematization Process by the theory of representations

Here, I would like to illustrate the mathematization process, as shown in Figure 9. Depending on what students have already learned, the focus of the symbol in Freudenthal's term 'life of living' differs. In the Gakkotosho textbook (2021 edition), the end-of-grade 2 section for 7-year-old children illustrates the process from I to II. It is illustrated by the theory of representation in Figure 12.

In the old theory, the model operates according to rules. For counting steps, a diagram is introduced as a sketch. Here, a diagram cannot be produced without manipulating the model in a manner that becomes the manner of translation from Model representation (I) to Figural representation (II). The next rule in b) is to draw a diagram 'step by step,' and not to skip, because finding the repetition is the objective. And the repetition is recognized by ignoring the largest ring. It is the moment that we feel troubled in c). And then, find properties. The property (III) implies the arithmetic operations d). It is from Figural representation (II) to Arithmetic-Algebraic representations (III). Here, the arithmetic recursive formula implies (I) and (II) as the rule to generate diagrams like copy and paste. The diagramming methodology has shifted from 'step by step' to repetition. It means the scale to see diagrams has changed. Finally, an alternative theory is established by mutual translation f). Under this new theory, the number of steps can be determined without manipulating the real Model.

This illustration is based on a Japanese second-grade primary school textbook. The approach changes several times: in the Tower Model (I), the view shifts at each tower step to notice the trouble between the 3-ring and 4-ring systems. In the Figural representation (II), the view changes from copying each model step to repeating and duplicating the previous diagram set.

The process from (II) to (III) might also be analyzed as a case of mathematization in upper school, when students learn to represent the sequence algebraically.

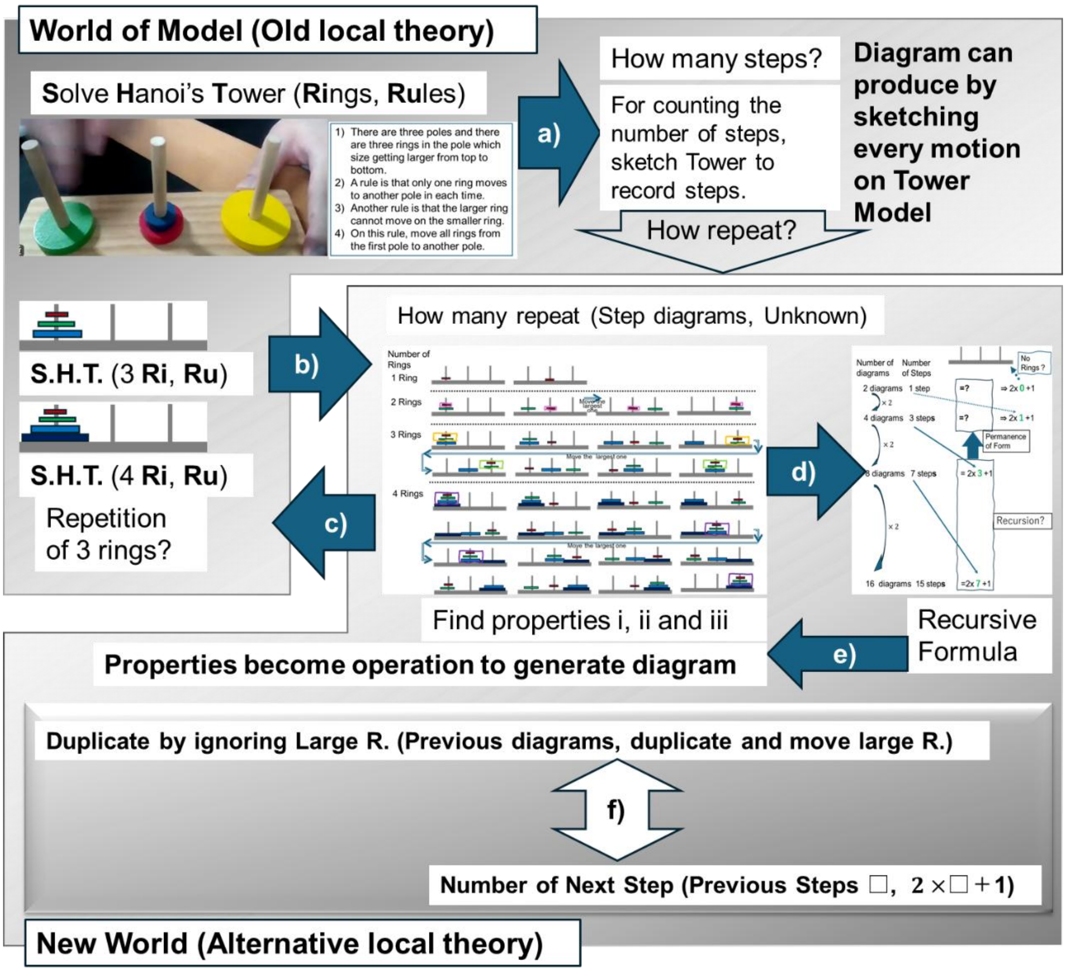


Figure 12. Mathematization on Hanoi's Tower

8. FINAL REMARKS

In the lecture, mathematical and computational thinking are emphasized, as are figural representations. It explained mathematization by using Newton's Fluxions (Colson, 1734). And representation theory was explained with additional examples on 'Rute Finding' in Somsaman et al. (2024, pp. 60-61) up to the Pascal Triangle. Due to time constraints, the theory of mathematical representations for mathematization was not explained in the lecture. This article clarifies it in relation to how it is seen. Ways of seeing are part of human character as well as of thinking mathematically. Mathematical representation is also theorized by the framework, and it is seen as. Even belief is also considered a human personality; educators should first explain it as an objective for teaching mathematics from the perspective of human character formation in education. On this necessity, the curriculum framework in Figure 2 was explained. Figures 3, 4a, and 4 b indicate contradictions that should be overcome in learning. Students overcome contradictions by and for themselves, which includes acquiring further or different ways to 'see it as' with mathematical representation. What students can use for thinking is what they have already learned. (I) and (II) on Figure 2 are also content of teaching, as well as knowledge and skills of (III). See it as discussed in the other terms, such as intuition, preoccupation, or bias, as a human perspective (see Hanson, 1969; Bachelard, 1934).

From the perspective of the Zone of Proximal Development, which explains the gap between what has been learned and what has not yet been learned but can be filled in by learners, intuition does not emerge from nothing. For filling in, see it as is also presupposed before the class, and to change or find a new way to see is the result of the class. These are theoretical explanations for designing the curriculum, teaching sequence, and implementation for human character formation.

In mathematics education research, the term ‘sense,’ such as symbol sense (Arcavi, 1994), has been similarly used for ‘see it as’ in this article, but its usage is more analytical in the context of informal reasoning. When we compare the usage of the senses, this article uses the term ‘see it as’ as part of the teaching content in the curriculum sequence, formally, as in Figure 4a and 4b in the Curriculum Framework on Figure 2. And it also adapts to the figural representations.

In the Era of New Math, Dienes (1978, p. 83), who proposed his level theory to address the mathematical structure in collaboration with J. Bruner, discussed mathematical abstraction as a process of encoding and decoding. In the Author’s representation theory, the term ‘translation’ describes the process by which the embedded symbol is generated in the old local theory, acquiring new self-generative operations to be transferred to the new local theory, as in Figures 9 and 12. Seeking and establishing new generative operations for the un-self-generative symbol is the key to mathematization. It does not always shift from concrete to abstract. It depends on what ‘see it as’ is possible for students and how new ‘see it as’ is learned. What’s necessary for a curriculum sequence is preparing representations that can be used for future learning, as well as applying what has been learned. See it as also necessary content when we teach representations.

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